Formulating a first-principles statistical theory of growing surfaces in two-dimensional Laplacian fields

Raphael Blumenfeld
Center for Nonlinear Studies and the Theoretical Division, Mail Stop B258, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

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A statistical theory of two-dimensional Laplacian growths is formulated from first principles. First, the area enclosed by the growing surface is mapped conformally to the interior of the unit circle, generating a set of dynamically evolving quasiparticles. Then it is shown that the evolution of a surface-tension-free growing surface is Hamiltonian. The Hamiltonian formulation allows a natural extension of the formalism to growths with either isotropic or anisotropic surface tension. It is shown that the curvature term can be included as a surface energy in the Hamiltonian that gives rise to repulsion between the quasiparticles and the surface. This repulsion prevents cusp singularities from forming along the surface at any finite time, and regularizes the growth. An explicit example is computed to demonstrate the regularizing effect. Noise is then introduced as in traditional statistical mechanical formalism, and a measure is defined that allows analysis of the spatial distribution of the quasiparticles. Finally, a relation is derived between this distribution and the growth probability along the growing surface. Since the spatial distribution of quasiparticles flows to a stable limiting form, this immediately translates into a predictability of the asymptotic morphology of the surface. An exactly solvable class of arbitrary initial conditions is analyzed explicitly.

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I. INTRODUCTION

Growing surfaces in diffusion controlled and generally Laplacian fields have been the focus of much attention recently. These processes are very simple to formulate but extremely difficult to analyze theoretically. Many examples of such growths can be found where the resulting morphologies are very ramified and generally exhibit a rich variety of patterns. Well-known cases are dendritic growth and solidification in supercooled liquid, diffusion-limited aggregation, electrodeposition, viscous fingering in Hele-Shaw cells, growth of bacterial colonies on an agar substrate, and many more [1]. The inherent difficulty in understanding these processes analytically stems from the characteristic instability of the moving boundary, combined with screening competition of the growing arms over the field. Consequently, the number of theoretical predictions that relate to such processes is surprisingly small compared to the large amount of existing phenomenological and numerical data.

There currently exist several variants of a renormalization group approach [2], and more recently two such generic approaches managed to yield rather accurate values for the scaling of the radius of gyration of several growths [3,4]. Generically, in such approaches an iterative procedure is carried out for the growth probability density. These approaches assume the existence of a limiting stable distribution with *scaling* properties and analyze the scaling exponent of the average mass with time or scale. However, a full theory that starts from the fundamental equations of motion (EOM) and leads in a step-by-step manner to predicting the full asymptotic structure of the surface, without those assumptions, is yet to be put forward.

Mostly due to conceptual and chemical simplicity, most of the literature treats growth of such patterns in two dimensions, and this discussion is no exception. I focus here on this case mainly because it allows for an elegant conformal formulation, which simplifies the formalism. Nevertheless, it should be stressed that the essential features that are relevant to the present theory can be applied to growth processes and evolving surfaces in higher dimensions, as will be shown elsewhere [5].

Already in the forties [6] and more recently [7] it has been proposed that in the case of two-dimensional growth of a surface-tension-free boundary in a Laplacian field, conformal mapping can be used to transform the problem to the dynamics of a many-body system. This approach converts the problem of solving a one parameter-dependent partial-differential equation (PDE) to that of solving a system of first order ordinary differential equations (ODE's), with each ODE corresponding to an EOM of one quasiparticle (QP) of the equivalent many-body system. This set of ODE's turns out to be strongly coupled and nonlinear, making the problem still very difficult to solve other than in special cases [7,8].

Thus this technique has found very little use in the research community. Moreover, this elegant description suffers from an even more acute problem. In most cases (i.e., for most initial conditions) the EOM break down after a finite time due to the inherent instability (of the Mullins-Sekerka type [9]) of the surface with respect to growth of perturbations on arbitrarily short lengthscales. In the absence of surface tension, these develop into cusp singularities along the physical surface, which correspond

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in the mathematical plane to zeros or poles of the conformal map arriving at the unit circle (UC) at a finite time. There have been some attempts to suppress this catastrophe by adding surface tension and using it to cut off the short lengthscales in a renormalizable manner [10]. These approaches, however, seem to be somewhat *ad hoc* in the sense that the procedure for cutting off the short scales can be arbitrarily chosen. In other words, one can introduce a phenomenological renormalizing procedure for the surface-tension-dependent term in the EOM of the physical surface, which readjusts the EOM of the singularities of the map and prevents the breakdown. The choice of the phenomenological term is arbitrary in a sense and renormalization approaches are known to introduce uncontrolled errors. A perturbative approach is also difficult because an arbitrarily small surface tension turns out to be a singular perturbation for the unregularized system, which, for a small surface tension, makes the surface's evolution very sensitive to initial conditions [11]. Another approach that has been proposed to prevent formation of such cusps relies on a recent observation [12] that tip splitting reduces local values of high surface curvature energy. It was therefore proposed that the mathematical quasiparticles (QP) split when they come too close to the surface, implying a field theoretical description of the problem.

Another issue in this context, which is significant for the purpose of the formalism presented here, is whether the system can be described by a Hamiltonian structure. It has been long known that this problem enjoys a set of conserved quantities [13,14], but the usefulness of these for integrating the EOM was not clear. This issue has been recently addressed by this author [15] and it appears that indeed the system enjoys a Hamiltonian formulation that under a given condition can even be integrated, as has been demonstrated for a specific family of initial conditions.

In this paper, I try to lay the foundations for a full theory of growth of such surfaces. The theory is constructed in five stages. (i) First the EOM of the surface, which is generally a first order PDE, is converted into a set of first order ODE's for quasiparticles of a many-body system, as mentioned above. (ii) A Hamiltonian structure of the system is formulated. (iii) Taking advantage of the existence of a Hamiltonian functional, I introduce the surface contribution as simply another energy term in the Hamiltonian. This term prevents occurrence of cusp singularities and makes the formulation valid for all times and for any initial condition. In the many-body Hamiltonian system, this term gives rise to an effective *repulsion* between the surface and the QP. This approach is suited to anisotropic, as well as isotropic surface tension. (iv) Next the master equation that governs the evolution of the spatial distribution of the QP is formulated. This distribution flows to a stable limiting form. At this stage, noise is naturally introduced into the theory in a fashion similar to traditional statistical mechanical theories. (v) The last step consists of translating the spatial distribution of the QP into the statistics of the growing surface, thus enabling us to analyze and predict from *first principles* the morphological features of the asymptotic pattern.

II. FORMULATION OF THE PROBLEM AND MAPPING INTO MANY-BODY DYNAMICS

The two-dimensional problem under study can be formulated as follows. Consider a simply connected line surface $\gamma(s)$ parametrized by an angular variable, $0 \le s < 2\pi$. On this boundary, the potential field Φ is fixed at a given value Φ_0. This field can represent an electrostatic potential, a concentration field for diffusion controlled processes, a thermal field, and much more. A higher potential value is assigned to a circular boundary of radius R_∞ that is much larger than the size of the area S enclosed by γ. Assuming no sources, the field outside S, Φ, is determined by Laplace's equation,

$$\nabla^2\Phi = 0 \ . \tag{2.1}$$

The surface is assumed to grow according to a constitutive rule that relates the local rate of growth proportionally to the local gradient of the field normal to the surface,

$$v_n = -\nabla\Phi \cdot \hat{\mathbf{n}} \ .$$

Being two dimensional, this process allows for an elegant use of complex analysis to write a closed form evolution equation [6,7,13]. First one maps the closed curve γ in the ζ complex plane conformally to the unit circle (UC) in a mathematical z plane:

$$\zeta = F(z) \ .$$

In the z plane, the complex potential field is simply

$$\Phi(z) = \ln(z/z_0) = \ln(|z|/|z_0|) + i[\arg(z) - \arg(z_0)] \ ,$$

where z_0 is some constant that is determined by the reference potential on the UC. So the complex field $\nabla\Phi$ along the physical surface is

$$-\nabla\Phi(\zeta) = -\left[\frac{\partial\Phi(\zeta)}{\partial\zeta}\right]^* = \frac{-i}{(zF')^*} \ , \tag{2.2}$$

where $*$ stands for complex conjugate and the prime indicates differentiation with respect to z. For the moving surface, $z \to e^{is}$ $(0 \le s < 2\pi)$, and therefore the actual curve evolves according to

$$\frac{\partial\gamma(s,t)}{\partial t} = -i\left[\frac{\partial\gamma^*(s,t)}{\partial s}\right]^{-1} \ . \tag{2.3}$$

Since this equation is obtained by monitoring only the normal velocity of the boundary at each s, it does not maintain the right parametrization with time. To correct this, one has to allow for a tangential velocity, so that a point can "slide" along the curve. This is essential for the purpose of studying the evolution of the surface through the dynamics of the map F. For the map to be conformal, both F and its inverse must be analytic in z outside the UC. But maintaining only the normal velocity spoils this analyticity because the components of the gradient of the potential field are not analytic. Therefore, if $\gamma(s,t)$ is to be described as the limit

$$\gamma(s,t)=\lim_{z\to e^{is}} F(z,t) ,$$

then the right hand side (rhs) of (2.3) needs to be augmented. The augmented evolution equation for the surface reads [7]

$$\partial_t \gamma(s,t) = -i\partial_s\gamma(s,t)\{|\partial_s\gamma(s,t)|^{-2} + ig[|\partial_s\gamma(s,t)|^{-2}]\} . \quad (2.4)$$

The first term on the rhs represents the field gradient normal to the surface as before. The second term is obtained through the demand that the rhs is analytic in z and represents the tangential component of the velocity, which causes the "sliding" of a point s along γ. Although mathematically important, this component has no real physical consequence for the advance of the surface since we can reparametrize the curve as we wish at each time step.

One can write now the EOM for the map F, which gives (2.4), but for the purpose of this presentation it is more convenient to write the EOM of $F'(z,t)\equiv dF(z,t)/dz$:

$$\partial F'/\partial t = \frac{\partial}{\partial z}\{zF'G\} . \quad (2.5)$$

The form of F' considered here is chosen to generally consist of a ratio of two polynomials. It turns out that one of the constraints on the map [12,15] is that these polynomials are of the same degree so that the map should preserve the topology far away from the growth, namely, that $\lim_{z\to\infty}F(z,t)=Az$, where A is a space-independent global scaling prefactor. This requirement amounts to leaving the original boundary conditions at R_∞ unchanged. The map now is

$$F'(z,t)=A(t)\prod_{n=1}^{N}\frac{z-Z_n}{z-P_n} . \quad (2.6)$$

The quantities $\{Z_n\}$ and $\{P_n\}$ represent the locations of the zeros and the poles of the map, respectively. As discussed below, the number of each species N may actually go to infinity by treating the local densities of these species, but for the sake of clarity, I will discuss in this presentation only discrete cases. Since the map and its inverse are conformal, then these poles and zeros must be confined to the interior of the UC that is mapped to the interior of the growth. Thus the evolution of the surface can now be expressed in terms of the dynamics of these zeros and poles. By inserting the explicit form of F' into (2.5), rearranging terms, and then comparing the residues on both sides of the resultant equation, we arrive at a set of first order ODE's for the location of the zeros and the poles [7,12]:

$$-A^2(t)\dot{Z}_n = Z_n\left\{G_0+\sum_{m'}\frac{Q_n+Q_{m'}}{Z_n-Z_{m'}}\right\} + Q_n\left\{1-\sum_m\frac{Z_n}{Z_n-P_m}\right\} \equiv f_n^{(Z)}(\{Z\};\{P\}) , \quad (2.7)$$

$$-A^2(t)\dot{P}_n = P_n\left\{G_0+\sum_m\frac{Q_m}{P_n-Z_m}\right\} \equiv f_n^{(P)}(\{Z\};\{P\}) ,$$

where

$$Q_n = 2\prod_{m=1}^{N}\frac{(1/Z_n-P_m^*)(Z_n-P_m)}{(1/Z_n-Z_m^*)(Z_n-Z_{m'})} ,$$

$$G_0=\sum_{m=1}^{N}\frac{Q_m}{2Z_m}+\prod_{m=1}^{N}\frac{P_m}{Z_m} ,$$

and where I have adopted the convention that the primed index indicates $m'\neq n$. From Eqs. (2.5)–(2.7), one can construct the evolution equation for $\ln A(t)$. In the following, I will disregard the evolution of this scale factor, which is unimportant to the main thrust of this presentation. The implication of this is that at each time step the growth is in fact rescaled such that the prefactor is always 1. The locations of these zeros and poles can now be considered as QP of a many-body system that follow the trajectories of Eq. (2.7).

Thus the problem of the growing surface has been transformed into the problem of the analysis of a many-body system. These general EOM have been analyzed in various limits [12] and for several particular initial conditions [8,10]. Such an analysis is not the purpose here. Rather, since the present aim is at a general theory, I now turn directly to formulate the Hamiltonian of the system.

III. HAMILTONIAN FORMULATION

The Hamilton form into which we wish to map the system,

$$H=H(\{J\};\{\Theta\}) ,$$

is generally a function of new canonical complex variables that depend on the coordinates $\{Z\}$ and $\{P\}$. First, let me convince the reader that there is ground for a belief in such a formulation. The first hint can be found in the EOM of the surface (2.3) This equation can be slightly changed to read

$$\frac{\partial\gamma(s,t)}{\partial t} = -i\frac{\delta s}{\delta\gamma^*(s,t)} , \quad (3.1)$$

where the spatial partial derivative has been replaced by the δ operator. This form, however, represents exactly Hamiltonian description if γ is interpreted as a field (complex) variable and s plays the role of an energy functional of γ [16]. Thus it appears that a Hamiltonian formulation does underlie the physical process. With this realization it is tempting to ask whether the addition of the tangential velocity makes a difference to this conclusion. To show that this is not so, consider the EOM (2.5). This equation can be interpreted as one of Hamilton's equations, where $\tilde{H}=zF'G$ is a Hamiltonian density and F' is one of the conjugate variables. Using the identity

$$\left[\frac{\partial F'}{\partial t}\right]\left[\frac{\partial t}{\partial z}\right]\left[\frac{\partial z}{\partial F'}\right] = -1 \quad (3.2)$$

and combining with Eq. (2.5) immediately yields the relation

$$\frac{\partial z}{\partial t}=-\frac{\partial(zF'G)}{\partial F'} . \tag{3.3}$$

Again we have arrived at a Hamiltonian description: Eqs. (2.5) and (3.3) constitute the conjugate pair of Hamilton's relations if z is interpreted as the variable that is formally conjugate to F'.

Translating the Hamiltonian back to the physical ζ plane $\tilde{H}(z)=H(\zeta)$, and rewriting the EOM for the surface in the original coordinates, we obtain

$$\frac{\partial}{\partial \zeta}H(\zeta)=\frac{\partial}{\partial t}\ln F'=\sum_{n=1}^{N}\frac{\dot{P}_n}{F^{-1}(\zeta)-P_n}-\frac{\dot{Z}_n}{F^{-1}(\zeta)-Z_n} . \tag{3.4}$$

Inspecting the EOM (2.7) immediately reveals that these are the exact direct consequences of (3.4) when a contour integral over $\partial H/\partial\zeta$ is taken around a close neighborhood of the location of the QP in the ζ plane. Hence it is the contour integration of H that connects surface dynamics to the many-body formulation. Therefore, since there is a Hamiltonian description that underlies the surface dynamics, it makes sense that the system of the $2N$ QP (i.e., of $4N$ degrees of freedom) is Hamiltonian too. The construction of a Hamiltonian directly to the many-body system has been recently carried out [15], where it has been shown that under a given condition the Hamiltonian is even integrable. Indeed, such integrability has been recently demonstrated explicitly for a class of arbitrary initial conditions. This class is generalized in the Appendix.

It should be remarked at this point that the above discussion suggests that the formulation presented here can be generalized to a *continuous* density of zeros and poles as follows. Inspection of the EOM and the signs of the residues in Eq. (2.5) shows that we can interpret the zeros and poles as positive and negative charges, respectively. Then the contour integrals in the plane that relate the map's evolution to the EOM of the charges can be regarded as Gauss integration around an area that contains some distribution of charges. Since all we know is the value of the integral over $\partial H/\partial\zeta$ it can be attributed to a continuous, rather than to a discrete, density of charges. The result will be a first order ODE for the *charge density* in this region. So by making the typical number of singularities within such an area very large, we effectively pass to the continuum limit. With this extension of the formalism, one can overcome quite a few difficulties that stem from the finiteness of the number of the singularities [17] and generally pass to a continuous field description of the problem.

IV. INTRODUCTION OF SURFACE TENSION

So far I have considered a free boundary (i.e., without surface tension) that evolves in an external Laplacian field. I now turn to discuss the effect that surface tension has on the growing process. Evidently, patterns that result from real growth processes do not entertain any cusps forming along the boundary. Depending on the system, this is either due to some microscopic atomistic cutoff scale, below which the above description ceases to apply, or because there is a macroscopic surface energy to be paid when the curvature of the surface increases. Here I focus on the second mechanism for two reasons: (i) The entire formulation presented here relies on the continuous aspect of the surface and therefore makes it cumbersome to treat atomistic cutoffs; (ii) many natural growth processes can be shown to enjoy a continuum description where the surface energy can be defined as a function of a continuous curvature, which is bounded along the surface.

Assuming then that there is a given surface energy that has a smoothening effect on the boundary, the question is what would this effect translate into in the context of the many-body system? To answer this question, one should first note that the radius of curvature that the QP enhances along the surface increases with the distance between a QP and the boundary, namely, the closer the QP is to the boundary, the sharper the distortion of the surface. The sign of this effect depends directly on the "charge" of the QP with protrusions corresponding to zeros and indentations to poles. For example, for a relatively isolated zero at

$$Z_0=(1-\delta)e^{is_0} , \quad \delta\ll 1 ,$$

the curvature at s_0 is $K(s_0)\sim 1/\delta^2$ (see Sec. V). Therefore, the effect of surface energy that prevents too small radii of curvature should be translated into preventing the QP from coming too close to the boundary. From this argument it follows that in the equivalent many-body Hamiltonian system the effect of a positive surface tension must correspond to *repulsion* between the QP and the inside of the boundary. It needs to be emphasized here that only because we have a Hamiltonian formulation available, the term "repulsion" can be used with any physical meaning. The Hamiltonian structure makes it possible to account for such effects in a natural energetic context, while without it surface effects could only be incorporated by introducing an additional *ad hoc* term into the EOM.

To prevent completely the cusps the repulsive potential between the surface and the QP must *diverge* as their separation vanishes. It follows that in this case we can consider the QP to be effectively confined to within a potential well that consists of an infinite wall (the surface boundary).

Example. To demonstrate how this method regularizes the growth let us consider the simple case discussed in the Appendix. The initial conditions of this growth process consist of N pairs of zeros and poles arranged symmetrically on N rays. The EOM for this system (A3) need to be augmented with the surface potential. The choice of the model potential is at our disposal at this stage (see a more detailed discussion on the surface potential below). For the purpose of the present example let us assume a form that gives rise to an arbitrary (negative) power α of the distance between the QP and the UC. The full EOM now become

$$-\frac{1}{N}\frac{\dot{x}}{x}=y/x-\mathcal{K}[1-2/N+(1+2/N)y/x]+\sigma_x/(1-x)^\alpha ,$$
$$-\frac{1}{N}\frac{\dot{y}}{y}=y/x-\mathcal{K}(1+y/x)+\sigma_y/(1-y)^\alpha , \tag{4.1}$$

where σ_x,σ_y are constants and the other notations are as in Eq. (A3). An analysis of these equations shows that the growth becomes uniform very quickly. In Fig. 1, I plot the resulting surface with and without the surface term for $\alpha=1$ and for three pairs of QP. Without the regularizing terms, Fig. 1(a) shows the formation of cusp singularities at $t=0.2673$ (arbitrary units). With the surface terms, the growth is observed to settle into a uniform process even at times that are orders of magnitude larger. To demonstrate the uniformity of growth, I rescale the area enclosed by the surface at each time step by $A(t)$, whereupon it can be seen that, asymptotically, the curves at different times collapse on top of each other. These processes have been run for times up to $t=1000$ in these arbitrary units to check that the asymptotic form is stable and does not change. Various different surface potentials have been found to produce very similar results. A general analysis of the dependence of the growth on the form of the functional properties of the surface potential is not intended here and is a subject for future research.

Thus with this regularization the above formalism of dynamics of singularities has been practically extended to hold up for $t\to\infty$, which has been heretofore one of the main disadvantages of this general approach. The only part that still needs to be clarified is whether the form of the map can still be described by a ratio of polynomials once the Hamiltonian is augmented by the surface term. It is this author's belief that this is so, albeit with the possibility of encountering a time-dependent, or even an infinite, number of singularities, depending on the form of the repulsive potential. A related approach has been considered recently, by Blumenfeld and Ball [12], where a surface term was introduced in the EOM. Only in this case, the proximity to the surface initiated production of opposite charges (poles and zeros in pairs) that acted to reduce the local curvature in front of an approaching zero. I will just remark here that that approach opens the door to a general interpretation of the field that is induced by the boundary and which is felt by the QP. In the presence of this field, particles can be spontaneously created by vacuum fluctuations and annihilated by encountering antiparticles. This interpretation fits quite naturally in the present formalism because a pole and a zero do indeed annihilate upon encounter, as mentioned already.

Let us now turn to discuss the form of the potential term in more detail. This term must be a functional of the curvature K, which, in turn, can be expressed in terms of the locations of the QP in the z plane [12]

$$K(s,\{Z\},\{P\})=\lim_{z\to e^{is}}|F'|^{-1}\left\{1+\mathrm{Re}\sum_{n=1}^{N}\left[\frac{z}{z-Z_n}-\frac{z}{z-P_n}\right]\right\} . \tag{4.2}$$

So if the Hamiltonian of the surface-free system was H_0, then the new Hamiltonian becomes

$$H_K=H_0+V(K(z,\{Z\},\{P\})) , \tag{4.3}$$

and the new EOM in the mathematical plane are derived from the new Hamiltonian as before:

$$\partial F'/\partial t=\frac{\partial}{\partial z}\{zF'G+V(K(z,\{Z\},\{P\}))\} . \tag{4.4}$$

Since the potential $V(K(z,\{Z\},\{P\}))$ should effect repulsion between the QP and the boundary, then the sign of V is immediately determined. For example, the simplest form that comes to mind for such a repulsive potential in the *mathematical plane* is

$$\mathrm{Re}\{V(K(z,\{Z\},\{P\}))\}=\sigma(z)[|F'|K(z,\{K\},\{P\})] . \tag{4.5}$$

The complex form of V can be found from the demand that this term is analytic outside the growth, which leads to

$$V=\frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint\frac{z+z'}{z+\epsilon-z'}\mathrm{Re}\{V\}\frac{dz'}{z'} . \tag{4.6}$$

It is important to note that the surface tension σ in this formulation can depend on z, therefore allowing for *anisotropic surface effects*, e.g., as in crystal growth. The reason for taking the potential term in the mathematical, rather than in the physical, plane is that it is there that the QP are moving and where they feel the effects of the "wall" along the UC. Note also that the form in (4.5) is easy to handle because it decouples naturally to a sum of individual contributions of the QP:

$$\mathrm{Re}\{V_0\}=\sigma(z) ,$$
$$\mathrm{Re}\{V(Z_n)\}=\sigma(z)\mathrm{Re}\frac{z}{z-Z_n} , \tag{4.7}$$
$$\mathrm{Re}\{V(P_n)\}=-\sigma(z)\mathrm{Re}\frac{z}{z-P_n} .$$

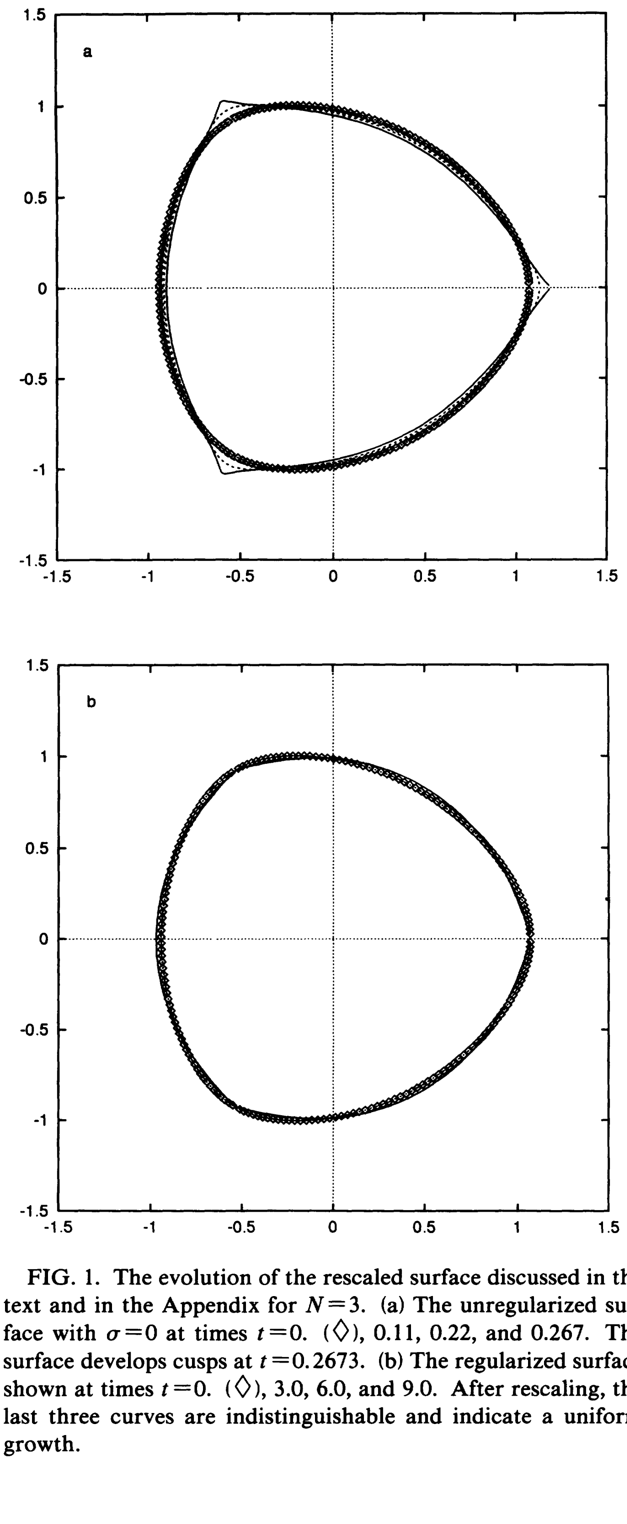

FIG. 1. The evolution of the rescaled surface discussed in the text and in the Appendix for $N=3$. (a) The unregularized surface with $\sigma=0$ at times $t=0.$ (◇), 0.11, 0.22, and 0.267. The surface develops cusps at $t=0.2673$. (b) The regularized surface shown at times $t=0.$ (◇), 3.0, 6.0, and 9.0. After rescaling, the last three curves are indistinguishable and indicate a uniform growth.

V. EFFECTS OF NOISE AND A STATISTICAL FORMULATION OF THE THEORY

The next, and probably technically the most difficult, step toward a theory of growth involves including the effect of noise. As is well known in growths governed by Laplacian fields, the patterns that such processes evolve into depend in a crucial way on the characteristics of the noise in the system. This noise can originate from many sources: general fluctuations in the local Laplacian field, discretization of the underlying background over which the field is solved (lattice growth), discretization of the incoming flux in the form of finite size particles that stick to the growing aggregate (e.g., off-lattice diffusion-limited aggregation, electrodeposition, and similar processes), etc. The noise can also be generally correlated in space and in time. In the present formulation, one way to incorporate all these effects is to interpret them as simply "smearing" the noiseless predetermined trajectories of the QP in the mathematical plane. The interpretation enjoys a new meaning now that we have a Hamiltonian available. The existence of a Hamiltonian immediately points to the existence of Liouville's theorem in this system, namely, that the distribution of the canonical variables in phase space is incompressible. Thus it is straightforward to write down an EOM for the time evolution of the distribution of the QP in phase space, and consequently it can be possible to analyze its asymptotic behavior. This exercise is currently being carried out by this author and will be reported at a later time. Either from such a calculation or via a phenomenological argument, one can devise a measure $\mu(H(\{Z\},\{P\}))$, (for example, the Gibbs measure $e^{-\beta H}$) and calculate *average* quantities weighted by this measure,

$$\langle X\rangle=\frac{1}{\mathcal{Z}}\int X\mu(H(\{Z\},\{P\}))d^NZd^NP\ , \tag{5.1}$$

where the partition function $\mathcal{Z}$ is

$$\mathcal{Z}\equiv\int\mu(H(\{Z\},\{P\}))d^NZd^NP\ .$$

Suppose that the Gibbs measure is indeed the relevant measure for this purpose. Then the Lagrange multiplier β, which in traditional statistical mechanics is associated with the temperature, would correspond here to the effective magnitude of the noise. This issue is also under current investigation. I should only comment that this approach should turn out to be equivalent to introducing noise directly in the EOM (2.7), and that such an equivalence should be possible to elucidate via an argument analogous to the fluctuation-dissipation theorem.

This formalism gives a well defined framework to describe the statistics of the QP and in general any property that depends explicitly on the distribution of their locations. It has been observed time and again that in many growth processes in Laplacian (and in other) fields, the growth probability along the surface seems to flow toward a stable asymptotic form. One manifestation of this phenomenon is the appearance of a time-independent multifractal function [18]. Since it is possible to show that the growth probability along the growing surface is directly related to the spatial distribution of $\{Z\}$ and $\{P\}$ [19], one can therefore analytically predict the statistics of the physical surface, its *asymptotic morphology*, and in particular the entire multifractal spectrum.

To illustrate how the above formalism is carried out, let me outline how to calculate the asymptotic (steady state) statistics of the curvature along the surface. It is shown below how knowledge of this distribution yields the asymptotic growth probability distribution, and how from the latter one can obtain the fractal dimension and the entire multifractal spectrum. There are two ways to go about such a calculation.

(i) Since the curvature is a function of the locations of the QP, one can simply calculate the nth moment of K from Eq. (5.1) by putting K^n for X. For example, if the Gibbs measure is assumed we have

$$M_n(z)\equiv\langle K^n(z,\{Z\},\{P\})\rangle$$
$$=\frac{1}{Z}\int K^n(z,\{Z\},\{P\})e^{-\beta(H_0+V)}d^NZd^NP \ . \quad (5.2)$$

This integral depends on z because the curvature is a local quantity, and in fact we are probing the statistics at a given location $z=e^{is}$. For isotropic growths, one can integrate the result over s (along the surface) for the final answer, but for anisotropic growths $[\sigma=\sigma(z)]$ expression (5.2) shows that the curvature statistics may well depend on the direction, which should not come as a big surprise.

(ii) The second approach is to first construct the master equation for the distribution of the locations of the QP, $\mathcal{P}$, by using Liouville's theorem:

$$\frac{\partial\mathcal{P}}{\partial t}+\sum_n f_n^{(R_n)}\frac{\partial\mathcal{P}}{\partial R_n}=0 \ , \quad (5.3)$$

where R_n is the nth component of the $4N$-dimensional ($2N$ degrees of freedom in two dimensions) vector $\mathbf{R}=(Z_1,\ldots,Z_N,P_1,\ldots,P_N)$. Since all existing observations of Laplacian growth processes indicate that the surface flows into a well-defined asymptotic morphological form with well-charted statistics, there must exist a stable limiting form that corresponds to the surface's statistics. Thus one gets a well-defined fractal dimension and a reproducible multifractal spectrum. We can therefore assume that there is a nontrivial steady-state solution where the direct dependence of $\mathcal{P}$ on t vanishes, which simplifies Eq. (5.3). Upon solution of this equation (clearly under some physically valid approximations, as is usually done in statistical mechanics) one obtains $\mathcal{P}(\{Z\},\{P\})$. Since the value of the local curvature K depends on the locations vector $\mathbf{R}$, one can consequently find the distribution function of K.

Example. Let us consider one possible approximation. Suppose that a QP, indexed 0, is located at $R_0e^{is_0}$ where R_0 is close to 1, and suppose that there is no other QP closer to the UC in the vicinity of s_0. This particular QP then dominates the local curvature at $z=e^{is_0}$ as can be seen from expression (4.2). Therefore, the approximation consists of completely ignoring the effect of the other QP at s_0. If one further assumes that the distribution of QP is isotropic, then the probability density of K, $\mathcal{P}_1(K)$, is simply defined in terms of the probability density of R_0, $\mathcal{P}_0(R_0)$, as follows:

$$\mathcal{P}_1(K)=\mathcal{P}_0(R_0)\left|\frac{dR_0}{dK}\right| \ . \quad (5.4)$$

With the above approximation, the curvature at s_0 becomes

$$K\approx C\frac{1+R_0}{(1-R_0)^2}\approx\frac{2C}{(1-R_0)^2} \ ,$$

where

$$C=\left|\prod_n(e^{is_0}-P_n)\Big/\prod_{m'\neq 0}(e^{is_0}-Z_{m'})\right|$$

is approximately constant for z in the neighborhood of s_0. Differentiating this expression with respect to R_0, inserting in (5.4), and expressing R_0 in terms of K gives

$$\mathcal{P}_1(K)=\text{const}\times\left|\frac{2C}{K}\right|^{3/2}\mathcal{P}_0\left|R_0=1-\left|\frac{2C}{K}\right|^{1/2}\right| \ . \quad (5.5)$$

This alternative route gives again an anisotropic z-dependent probability density of K as discussed above.

A significant point to note in this simple calculation is that even if $\mathcal{P}_0$ possesses exponentially decaying tails, the probability density of K decays *algebraically.* This feature is the signature of scale invariant and fractal structures. Therefore, already this crude approximation gives us a hint regarding the origin of the fractality observed in related real growth processes, such as electrodeposition, diffusion-limited-aggregation, solidification, bacterial growth, etc.

It is possible to generalize the above calculation to a multivariate distribution (i.e., using the general dependence of K on all the R_n's) without using the above approximation. This is outside the scope of this paper and will be discussed elsewhere. In any case, the above demonstrates how to use the knowledge of the distribution of the locations of the QP to determine the distribution of the curvature. The latter comprises, by definition, the morphology of the growing surface, which can now be uniquely determined. Thus the statistical formulation presented here constitutes the beginning of a full theory for the problem of a surface growing in a Laplacian field.

VI. DISCUSSION AND CONCLUDING REMARKS

To conclude, I have formulated here an initial theory for growth of surfaces in a two-dimensional Laplacian field. This has been carried out in several stages. First, following previous results, the evolution of the surface has been transformed into the problem of studying the dynamics of a many-body system. The quasiparticles (QP) in this system consist of N zeros and N poles of the conformal map that maps the interior of the growth onto the area enclosed by the unit circle (UC). In passing, I have pointed out how the formalism can be extended to include an infinite number of QP and subsequently to describe a continuous density of these particles. The next step was to show that the growth process is governed by Hamiltonian dynamics and to explicitly write down this Hamiltonian. Whether the Hamiltonian is integrable or not is not directly relevant to the theory formulated here, but a general class of arbitrary initial conditions that result in integrable dynamics has been explicitly analyzed and solved for in the Appendix. The existence of a Hamiltonian that underlies the dynamics immediately opens new horizons, which I exploited for the construction of the theory. The first difficulty that the Hamiltonian helps to overcome is the inherent instability of this general growth problem. By incorporating the surface effect directly in the Hamiltonian as a potential term, formation of cusp singularities along the surface is eliminated and the validity of the equations of motion (EOM) is extended to infinite time. I have argued that the potential term must correspond to a repulsive interaction between the QP and the surface. Since this potential diverges as a QP approaches the surface, the system is then confined to a well with infinite walls located on the UC. An explicit example that demonstrates the regularization by this method has been computed and plotted. This approach is naturally suited to both isotropic and anisotropic surface-tension effects, which heretofore could only be included in an *ad hoc* manner by assuming an extra term in the EOM of the surface. The present formalism can easily enjoy a field formulation in the sense that (i) QP can annihilate as can be seen directly from the form of the map (2.6) and (ii) depending on the nature of the field that the QP's move in, they can split. This latter feature has been suggested and used previously [12] with the observation that splitting of zeros is a mechanism that reduces locally high curvatures. In the physical growth, such a split corresponds either to tip splitting or to side branching, depending on the details of the process.

Turning to the statistics of the growth, let us first recall what we require from a full theory of growth. The theory should start from the basic EOM of the surface and, taking into account the noise that affects the growth process, should predict statistical properties of the asymptotic morphology that the surface evolves into. This is based of course on the observations that such an asymptotic morphology does exist in most Laplacian growths, e.g., in diffusion-limited aggregation, solidification, electrodeposition, bacterial growth, and others. But what does one mean by "morphology" in this context? It is only recently that a quantitative definition of this vague concept has been proposed for scale-invariant structures [20]. Note, however, that knowledge of the distribution of the curvature along the surface is equivalent to knowledge of the morphology of the asymptotic structure. For example, in the present context a popular measurable quantity is the dimension of the growth that relates to the time dependence of the size of the growth (the size can be defined by the radius of gyration for an aggregate or by the equivalent circular capacitor [21]). This quantity relates directly to the third moment of the growth probability distribution [22]. A customary generalization has been to study higher moments of this distribution. These can be cast in one function termed the multifractal function (or spectrum). Although this author believes that the multifractal function lacks sensitivity for useful characterization of the surface's morphology for these growth processes, it is nevertheless a signature of the structure. Therefore, one acid test of the present theory is whether it can lead to a prediction of the fractal dimension in particular and this entire function in general. In fact, the observation in Sec. V that the curvature entertains a power-law distribution even for well-behaved distributions of the QP's already points to the origin of scale invariance and fractality. Thus this formalism has the potential to *derive* the onset of such behavior (rather than assume scale invariance, as is usually done in the literature for analytical calculations). Indeed, full knowledge of the curvature statistics is sufficient to determine these quantities as follows. The local curvature of the evolving surface can be related to the local field gradient normal to the surface. The latter can be related in these processes through some constitutive relation to the growth probability p. For example, a popular such relation is

$$p=|\nabla\Phi|^{\eta}\sim K^{\eta} , \tag{6.1}$$

with η a parameter that can be adjusted according to the system under consideration. Thus from the knowledge of $\mathcal{P}_1(K)$ it should not be too difficult to derive the distribution of p and therefore the entire multifractal function.

To facilitate such a calculation, the next stage consists of employing the existence of Liouville's theorem due to the Hamiltonian description and considering the spatial distribution of the QP. This was carried out in two ways.

(i) First, considering an ensemble of initial conditions, one defines a measure $\mu(H)$, from which the partition function Z can be calculated. Writing the explicit dependence of the curvature along the surface on the locations of the QP, one can then calculate directly the moments of the probability density of the curvature. As in the case of the Gibbs measure $\mu(H)=e^{-\beta H}$, one can identify a Lagrange multiplier β that corresponds to smearing of the trajectory due to fluctuations in the process. This introduction of noise is analogous to (but seems to this author somewhat more natural than) introducing noise effects directly in the EOM of the QP. Nevertheless, this equivalence can be probably demonstrated by an analogue of the usual fluctuation-dissipation theorem, an exercise that has not been attempted in this presentation.

(ii) The second is a more dynamical approach. The existence of a Hamiltonian implies by Liouville's theorem that the volume occupied by the system in the $4N$-dimensional phase space is incompressible. Combining this with the observation that the statistics of the surface flows toward a stable fixed point immediately leads to the simplified master equation (5.3) for the distribution of the locations of the QP in the mathematical plane, I have not attempted here to solve this equation in any limit or approximation. Rather, I showed that the solution to that master equation yields all the needed information on the asymptotic morphology of the evolving surface, again by using the explicit dependence of the curvature on the locations of the QP.

Thus the curvatures statistics can be calculated in the present approach, and therefore the entire multifractal function as well as other relevant quantities. For example, combining the approximation (5.5) with the constitutive relation (6.1), one can relate the growth probability distribution $\mathcal{P}_2(p)$ to the distribution of the QP as follows:

$$\mathcal{P}_2(p)=\text{const}\times p^{-1-1/2\eta}\mathcal{P}_0(1-(2C)^{1/2}p^{-1/2\eta}) . \tag{6.2}$$

Again, the power-law decay of this probability, which appears *regardless of the behavior of* $\mathcal{P}_1$, *points towards the* possible origin of fractality in the system.

Thus, to the best of this author's knowledge, this theory represents currently the only approach that can lead to a quantitative calculation of all these properties from first principles.

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APPENDIX: AN INTEGRABLE CASE STUDY OF *N*-SYMMETRIC GROWTH WITH ARBITRARY INITIAL CONDITIONS

In this Appendix, I consider a particular class of arbitrary initial conditions for which the system is integrable and the solution to the set of EOM can be obtained explicitly. Let us consider an initial surface that is represented at $t=0$ by the form

$$\gamma(s,0)=e^{is}+\sum_{n=1}^{N} R_n \ln[e^{is}-P_n(0)] , \tag{A1}$$

where

$$R_n=\prod_{m'\neq n}^{N} \{[P_n(0)-Z_{m'}(0)]/[P_n(0)-P_{m'}(0)]\} .$$

It can be shown that the surface-tension-free propagating curve $\gamma(s,t)$ can also be described by this form for any later time if the values of P_n and Z_n are substituted by their time-dependent values. The form (A1) is valid for any number of QP, where R_n should be interpreted as the residues of the function $F'=dF/dz$ when a contour integral is taken around the nth pole P_n. Moreover, one can easily convince oneself from the EOM [7] that the number of QP, N, of each kind is invariant under the EOM. Thus the growth problem consists now of investigating the dynamics of N zeros at $Z_n(t)=Z(t)e^{in\alpha}$, where $\alpha\equiv 2\pi/N$ and N poles at $P_n(t)=P(t)e^{in\alpha}$[$Z(t)$ and $P(t)$ are real functions of time]. The initial values $Z(t=0)$ and $P(t=0)$ are completely arbitrary as long as $P(0)$, $Z(0)\neq 0$, and $P(0)\neq Z(0)$. The dynamics of these singularities can be found by substituting directly into Eqs. (2.7). An observation that is worthwhile to note is that both from symmetry arguments and from direct analysis of the EOM, one can see that the motion of all the QP will be purely radial. Therefore, the arguments $n\alpha$ stay constant and the only time-dependent variables are the radial locations from the origin $Z(t)$ and $P(t)$. Next we observe that requiring that F be holomorphic outside γ implies that there is a sum rule imposed on the locations of the singularities [12,15]

$$\sum_{n=1}^{N} Z_n(t)=\sum_{n=1}^{N} P_n(t) , \tag{A2}$$

which can be shown to be identical to requiring that

$$\sum_{n=1}^{N} R_n=0 .$$

These constraints simplify the EOM that can now be written as

$$-\frac{1}{Nx}\dot{x}=y/x-\mathcal{H}[1-2/N+(1+2/N)y/x] ,$$
$$-\frac{1}{Ny}\dot{y}=y/x-\mathcal{H}(1+y/x) , \tag{A3}$$

where I have defined $x\equiv Z^N$, $y\equiv P^N$, and $\mathcal{H}\equiv(1-xy)/(1-x^2)$. This system of equations displays a qualitatively different behavior when $P(0)$ is smaller or larger than $Z(0)$. In the first case, cusp singularities appear at a finite time that corresponds to the time when a zero arrives at the UC. In the second case, such singularities are avoided for any finite time. The issue of particular cuspless growth solutions due to specialized choices of the initial conditions has received some attention in the literature [23]. This issue is not directly relevant to the main thrust of this paper, but it should be pointed out that recent calculations [24] for the present N-symmetric case suggest that the noncuspiodal solution for $P(0)>Z(0)$ is unstable for small perturbations in α, under which it flows into a cusp-forming solution.

To emphasize a point made in the text, I intentionally choose initial conditions that lead to cusp formation, $P(0)<Z(0)$ and $\arg(P_n)=\arg(Z_n)$. The point is that regardless of whether or not the solution breaks down after a finite time, the dynamics is Hamiltonian up to the moment of breakdown. In other words, the dynamics remain Hamiltonian (and for the present case, integrable) as long as the EOM are valid. When cusps form, the very EOM cease to be valid and the entire framework of transforming the growth problem into the many-body dynamics no longer holds. Indeed, one of the goals in the text is to extend the formalism with the aid of surface energy so as to make it hold for $t\to\infty$ *regardless of initial conditions.* Since this has been achieved in the text, it makes no difference to the present analysis whether the description of the surface-tension-free curve holds for a finite or infinite time.

Inspection of the EOM shows that if the zero and the pole meet, say, at r_0, they keep moving together at an exponential rate,

$$r(t)=r_0 e^{Nt} .$$

This observation, however, is academic because once a pole and a zero meet, they "annihilate." This can be verified by inspecting the form of the conformal map (2.6), upon encounter, $P_n=Z_n$, and therefore the corresponding terms $(z-Z_n)/(z-P_n)$ cancel and these QP disappear from the scene.

Using (A1) and a straightforward manipulation yields for the explicit form of the interface at any time $t>0$

$$\gamma(s,t)=e^{is}-\frac{Z(t)(1-y/x)}{N} \times\sum_{n=1}^{N} e^{2n\pi i/N}\ln[e^{is}-P(t)e^{2n\pi i/N}] . \tag{A4}$$

Carrying out a rather tedious manipulation of the EOM (A3), one can show that the following is a constant of the motion

$$\frac{y^{1-2/N}}{x-y}+L(y)=\text{const} , \tag{A5}$$

where

$$L(y)\equiv\frac{1}{N}\int^{y^2}\frac{u^{-1/N}du}{1-u} . \tag{A6}$$

The EOM can be integrated out now and the trajectories of P and Z can be found explicitly.

We can now find the canonical action-angle variables in terms of the original coordinates. The action variable can be immediately set to

$$J=\frac{y^{1-2/N}}{x-y}+L(y) , \tag{A7}$$

which we know is a constant of the motion. The Hamiltonian is then

$$H=\omega J ,$$

with ω some constant frequency and with the angle variable $\Theta=\omega t+\Theta_0$. The fact that we have only one action and one angle variables reflects the degeneracy of the problem due to the N-rotational symmetry, where only $Z(t)$ and $P(t)$ remain the relevant degrees of freedom. Thus we have an integrable Hamiltonian that depends only on half the number of degrees of freedom J. Substituting x from Eqs. (A5) or (A7) into the second equation of the set (A3) yields immediately the result for $y(t)$ in the form

$$t-t_0=\frac{1}{N}\int^{y}\frac{1-\xi^2(1+\mathcal{L})^2}{\xi(1-\xi^2)}d\xi , \quad \mathcal{L}(y)\equiv\frac{y^{-2/N}}{J-L(y)} . \tag{A8}$$

And substituting this into the first of the equations (A3) gives the corresponding solution for $x(t)$.

As mentioned in the text, in this treatment the prefactor in front of the conformal map F, $A(t)$, is taken to be unity. Since an implicit assumption in the general formulation of the problem is that the flux into the growth is constant in time, then the total area enclosed by the surface should increase linearly with time. Thus by maintaining a unity prefactor, the growth is in fact rescaled at each time step by $1/A^2(t)$. For this reason, a plot of this surface will reveal sections of the boundary that seem to retreat with time, although the actual physical surface always grows outwards. The evolution of this prefactor follows a first order ODE as has already been discussed in the text.

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